Review: Newton’s Laws

1\textsuperscript{st} Law: A body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.

2\textsuperscript{nd} Law: The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

3\textsuperscript{rd} Law: When a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

Newton’s 2\textsuperscript{nd} Law

\[ \bar{F} = m \ddot{a} = m \frac{d\bar{V}}{dt} \]

- Linear Momentum = mV
- \textbf{Restated:} The rate of change of the momentum of the body is equal to the net force acting on the body.
- \textbf{Conservation of Momentum:} when the net force acting on the object is zero, momentum is conserved
  - \textit{Why do we care?} Useful when analyzing collisions…

Conservation of Angular Momentum

- Analogous to Conservation of Linear Momentum
- Used for rotating, rigid bodies

\[ M = I \alpha = I \frac{d\dot{\omega}}{dt} = \frac{dH}{dt} \]

Where : 
\[ M = \text{net torque applied to body} \]
\[ I = \text{moment of inertia of the body about the axis of rotation} \]
\[ \dot{\omega} = \text{angular acceleration} \]
\[ \frac{dH}{dt} = \text{rate of change of angular momentum} \]

Angular Momentum about x-axis

- Scalar form for a rigid body rotating about the x-axis

\[ M_x = I_x \frac{d\omega_x}{dt} = \frac{dH_x}{dt} \]
Conservation of Angular Momentum Principle

- The rate of change of the angular momentum of a body is equal to the net torque acting on it.

\[ \vec{M} = \frac{d\vec{H}}{dt} \]

- Used to describe skaters spinning faster when they draw their arms in.

Reynolds Transport Theorem

- A moving system flows through a fixed control volume.
- The moving system transports extensive properties across the control volume surfaces.
- Need a bookkeeping method to keep track of the properties that are being transported into and out of the control volume.

Control Volume Conservation Equation

- Conservation of mass (for all species)
- Newton’s 2nd law of motion (momentum)
- First law of thermodynamics (energy)

Application of Reynolds Transport Theorem
Summary

• Reynolds Transport Theorem can be applied to a control volume of finite size
  – We don’t need to know the flow details within the control volume!
  – We do need to know what is happening at the control surfaces.
• We will use Reynolds Transport Theorem to solve many practical fluids problems

Linear Momentum Equation

\[ \sum \vec{F} = \rho \frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt} = \frac{d}{dt} \int \vec{V} \rho dV \]

Reynolds Transport Theorem Applied to Linear Momentum:

\[ \sum \vec{F} = \frac{d}{dt} \int \rho \vec{V} dV + \int \rho \vec{V} (\vec{V} \times \hat{n}) dA \]

Applications

• Steady Flow \( \sum \vec{F} = \rho Q (\vec{V}_{out} - \vec{V}_{in}) \)
  – One Inlet, One Outlet
  \[ \sum F_x = \rho Q (V_{out,x} - V_{in,x}) \]
  \[ \sum F_y = \rho Q (V_{out,y} - V_{in,y}) \]
  \[ \sum F_z = \rho Q (V_{out,z} - V_{in,z}) \]

NOTE: FE Fluids

• Within the previous development is the application of dot and cross product.
• We did not do this because my goal is for you to apply these relationships to real world problems….
• Other Applications:
  – Jet Propulsion (FE Pg. 48)
  – Deflectors and Blades
    • Fixed Blade (FE Pg. 49)
    • Moving Blade (FE Pg. 49)
    • Impulse Turbine (FE Pg. 49)

REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

- Motion of a rigid body is a combination of:
  - Translation from its center of mass
    - Analyzed using linear momentum
  - Rotation about its center of mass
    - Analyzed using angular distance $\theta$, angular velocity $\omega$, and angular acceleration $\alpha$.

Moment or Torque

- From Newton’s 2nd law, angular acceleration ($\alpha$) is caused by a force acting in the tangential direction.
- **Moment (M) (torque):** strength of the rotating effect, is proportional to the magnitude of the force and its distance from the axis of rotation.
- **Moment Arm:** the perpendicular distance from the axis of rotation to the line of action of the force.
**Moment or Torque**

- Total torque (moment) acting on a rotating rigid body about an axis:

\[
M = \int r^2 \alpha \, dm = \left[ \int r^2 \, dm \right] \alpha = I \alpha
\]

- Where:
  - \( I \) is the moment of inertia of the body about the axis of rotation
  - measure of the inertia of a body against rotation.
  - Note: rotational inertia of a body depends on the distribution of the mass of the body with respect to the axis of rotation.

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**Analogy between corresponding linear and angular quantities.**

- Mass, \( m \) \( \rightarrow \) Moment of inertia, \( I \)
- Linear acceleration, \( \ddot{a} \) \( \rightarrow \) Angular acceleration, \( \ddot{\alpha} \)
- Linear velocity, \( \vec{V} \) \( \rightarrow \) Angular velocity, \( \vec{\omega} \)
- Linear momentum \( m \vec{V} \) \( \rightarrow \) Angular momentum \( I \vec{\omega} \)
- Force, \( \vec{F} \) \( \rightarrow \) Torque, \( \vec{M} \)
- \( \vec{F} = m \vec{a} \) \( \rightarrow \) \( \vec{M} = I \vec{\alpha} \)
- Moment of force, \( \vec{M} \) \( \rightarrow \) Moment of momentum, \( \vec{H} \)
- \( \vec{M} = \vec{r} \times \vec{F} \) \( \rightarrow \) \( \vec{H} = \vec{r} \times m \vec{V} \)

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**Angular momentum**

- **Angular Momentum**, of a point, mass, \( m \), about an axis
  - AKA: moment of momentum, called the can be expressed as
  - Total angular momentum of a rotating rigid body is:

\[
H = \int \vec{r} \times \omega \, dm = \left[ \int \vec{r} \times \omega \, dm \right] = I \vec{\omega}
\]

- The vector form of angular momentum is:

\[
\vec{M} = I \vec{\alpha} = I \frac{d\vec{\omega}}{dt} = \frac{d(I \vec{\omega})}{dt} = \frac{d\vec{H}}{dt}
\]

- Note: angular velocity is the same at every point
- Rate of change of angular momentum, is:
Shaft power

• The angular speed of rotating machinery is typically expressed in rpm and denoted by $\hat{n}$.
• The angular velocity of rotating machinery is:

$\omega = 2\pi \hat{n}$

$W_{\text{shaft}} = \omega M = 2\pi \hat{n}M$

Rotational kinetic energy

• The rotational kinetic energy of a body of mass $m$ at a distance $r$ from the axis of rotation is

$$KE = \frac{1}{2}mr^2\omega^2$$

• The total rotational kinetic energy of a rotating rigid body about an axis can be determined by

Centripetal acceleration and force

• During rotational motion, the direction of velocity changes even when its magnitude remains constant.
• The centripetal acceleration changes the direction of the velocity. Its magnitude is

$$a_r = \frac{V^2}{r} = r\omega^2$$

• Centripetal acceleration is directed toward the axis of rotation. The centripetal force, which induces the acceleration, is

• Tangential and radial accelerations are perpendicular to each other, and the total linear acceleration is determined by their vector sum,

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

ANGULAR MOMENTUM EQUATION

• Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them, which are best analyzed by the angular momentum equation,

• The moment of a force $\vec{F}$ about a point $O$ is the vector (or cross) product.

• Whose magnitude is

$$M = Fr \sin \theta$$
ANGULAR MOMENTUM EQUATION

- The direction of the moment vector \( \vec{M} \) is determined by the right-hand rule.
- Replacing the vector \( \vec{F} \) by the momentum vector \( \vec{p} \) gives the moment of momentum, also called the angular momentum \( \vec{H} = \vec{r} \times \vec{p} \).
- The angular momentum of a differential mass \( dm \) is \( (\vec{r} \times \vec{V}) \rho \, dV \).

**Moment of momentum (system):**

\[
\frac{d\vec{H}}{dt} = \frac{d}{dt} \int \left( \vec{r} \times \vec{V} \right) \rho \, dV
\]

THE ANGULAR MOMENTUM EQUATION

- Rate of change of moment of momentum:
  \[
  \frac{d\vec{H}}{dt} = \frac{d}{dt} \int \left( \vec{r} \times \vec{V} \right) \rho \, dV
  \]
- The rate of change of angular momentum of a system is equal to the net torque acting on the system (valid for a fixed quantity of mass and an inertial reference frame).

THE ANGULAR MOMENTUM EQUATION - Special Cases

- Steady Flow:
  \[
  \sum \vec{M} = \int_{CS} \left( \vec{r} \times \vec{V} \right) \rho (\vec{V} \cdot \hat{n}) \, dA
  \]
  - In many practical applications, an approximate form of the angular momentum equation in terms of average properties at inlets and outlets becomes
  \[
  \sum \vec{M} = \frac{d}{dt} \int_{CV} \left( \vec{r} \times \vec{V} \right) \rho \, dV + \int_{CS} \left( \vec{r} \times \vec{V} \right) (\vec{V} \cdot \hat{n}) \, dA
  \]
  - No correction factor is introduced since it varies from problem to problem and the induced error is small.
- Steady Flow
  \[
  \sum \vec{M} = \sum_{out} \vec{r} \times \vec{mV} - \sum_{in} \vec{r} \times \vec{mV}
  \]
Flow with No External Moments

- When there are no external moments applied, the angular momentum equation reduces to

\[
0 = \frac{dH_C}{dt} + \sum_{\text{in}} \hat{r} \times \vec{m}\vec{V} - \sum_{\text{out}} \hat{r} \times \vec{m}\vec{V}
\]

- When the moment of inertia \( I \) of the control volume remains constant, then

\[
\vec{M} = I \ddot{\vec{a}} = \sum_{\text{in}} (\hat{r} \times \vec{m}\vec{V}) - \sum_{\text{out}} (\hat{r} \times \vec{m}\vec{V})
\]

EXAMPLE: Bending Moment Acting at the Base of a Water Pipe

- Underground water is pumped to a sufficient height through a 10 cm diameter pipe that consists of a 2 m long vertical and 1m long horizontal section. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point A) and the required length of the horizontal section that would make the moment at point A zero.

Radial-Flow Devices

- Flow in the radial direction normal to the axis of rotation and are called radial flow devices.
- In a centrifugal pump, the fluid enters the device in the axial direction through the eye of the impeller, and is discharged in the tangential direction.

Radial-Flow Devices

- Consider a centrifugal pump. The impeller section is enclosed in the control volume.
- The average flow velocity, in general, has normal and tangential components at both the inlet and the outlet of the impeller section.
- when the shaft rotates at an angular velocity of \( \omega \), the impeller blades have a tangential velocity of \( \omega r \), at the inlet and \( \omega r_2 \) at the outlet.
Radial-Flow Devices

- The conservation of mass equation tells
  \[ \dot{V}_1 = \dot{V}_2 = \dot{V} \rightarrow \]
  
- where \( b_1 \) and \( b_2 \) are the flow widths at the inlet and outlet.
- Then the average normal components are:
  \[ V_{1,n} = \frac{\dot{V}}{2\pi r_1 b_1} \quad \text{and} \quad V_{2,n} = \frac{\dot{V}}{2\pi r_2 b_2} \]

EXAMPLE: Power Generation from a Sprinkler System

- A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

Radial-Flow Devices

- The normal velocity components and pressure act through the shaft center and contribute no torque. Only the tangential velocity components contribute to the angular momentum equation, which gives the famous Euler's turbine formula:
  \[ T_{\text{shaft}} = \dot{m}(r_2 V_{2,t} - r_1 V_{1,t}) \]
  \[ T_{\text{shaft}} = \dot{m}(r_2 V_{2} \sin \alpha_2 - r_1 V_{1} \sin \alpha_1) \]
- In the idealized case, \( T_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2) \)
- The shaft power

Sprinkler Rotation

- When water is squirted out of a sprinkler, the sprinkler head rotates in the direction opposite to the water flow (called the normal direction) due to the reaction force on the head.
EXAMPLE: Power Generation from a Sprinkler System

- Discussion of two limiting cases

When sprinkler is not rotating, \( V_r = 63.66 \text{ m/s} \) and \( T_{shaft} = 764 \text{ Nm} \), but power is zero because shaft is not rotating.

When sprinkler is rotating, but disconnected from the generator, \( V_r = 0 \) and \( V_{jet} = V_{sprinkler} = 63.66 \text{ m/s} \) and shaft rotates at 1013 rpm, but torque and shaft power are zero.